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Einstein–Maxwell Null Fields with a Null Current Distribution

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Abstract

Null Einstein–Maxwell charge-free fields such that the propagation vector is a scalar multiple of a gradient are not determined uniquely by the geometry of the space-time. The metric for space-times admitting such exceptional fields can always be transformed to the Wyman–Trollope form. This same result follows if there is a non-vanishing null current density associated with the field.

1. *Introduction*

The Rainich–Misner–Wheeler (1925, 1957) conditions such that a space-time admits an electromagnetic field with zero charge and current density are

$$R_\alpha{}^\alpha = 0, \qquad R_{\alpha\beta} V^\alpha V^\beta < 0$$

$$R_{\alpha\beta} R^{\beta\tau} = \tfrac{1}{4}(R_{\gamma\beta} R^{\gamma\beta})\, \delta_\alpha{}^\tau, \qquad \theta_{\mu;\tau} = \theta_{\tau;\mu}$$

where V^α is any arbitrary time-like vector and

$$\theta_\mu = i(R_{\tau\beta} R^{\tau\beta})^{-2} \sqrt{(g)} . \epsilon_{\mu\alpha\rho\gamma} R^\rho{}_\sigma R^{\sigma\gamma;\alpha}$$

θ being the complexion of the electromagnetic field, and $\epsilon_{\mu\alpha\rho\gamma}$ is the usual antisymmetric permutation tensor, having the value +1 if μ, α, ρ, γ is an even permutation of 0, 1, 2, 3, and −1 if odd. These conditions have been extended to the case of a non-zero massless current distribution (Goodinson & Newing, 1968). In both cases, the conditions are found to break down when the field is null, since $R_{\alpha\beta} R^{\alpha\beta}$ is then zero. Several authors have considered vacuum null fields, and such fields are found to be determined uniquely by the metric of the space-time, except in the special case when the propagation vector of the null electromagnetic field L_α satisfies

$$\epsilon^{\alpha\beta\gamma\delta} L_{\beta;\gamma} L_\delta = 0 \tag{1.1}$$

and is therefore expressible as a scalar multiple of a gradient. When (1.1) is satisfied, Peres's (1961) exceptional case occurs, and Geroch (1966) has discussed the form of the metric for the class of space-times when this is so, i.e. the case for which the metric does not determine the electromagnetic field uniquely. He suggested that a coordinate system existed for which the metric had the standard form,

$$ds^2 = \alpha\, dx^{1^2} + 2\, dx^0\, dx^1 + 2\, dx^1(\beta\, dx^2 + \gamma\, dx^3) - (dx^{2^2} + dx^{3^2})$$

where α, β, γ are independent of x^0. Space-times for which the propagation vector L_α is equal to a scalar multiple of a gradient have been discussed by Wyman & Trollope (1965), who obtained

$$ds^2 = \alpha\, dx^{1^2} + 2\, dx^0\, dx^1 + 2\, dx^1(\beta\, dx^2 + \gamma\, dx^3) - k(dx^{2^2} + dx^{3^2})$$

as the standard form for the metric. A particular case of a general solution obtained by Wyman & Trollope (1965) is

$$ds^2 = 2\alpha_1\, dx^{1^2} + 2\, dx^1\left(dx^0 - \frac{x^0}{x^2}\, dx^2\right) - \frac{1}{\sqrt{x^2}}(dx^{2^2} + dx^{3^2})$$

where $\alpha_1 = 2x^{2^2}$.

It does not appear to be possible to convert this latter metric into the Geroch form and therefore it seems worthwhile to investigate the general form of metrics when Peres's exceptional case occurs.

Also it will be shown that the metric for any space-time admitting a null field with a null current can be expressed in the Wyman–Trollope form.

2. *Electromagnetic Equations in Terms of a Null Tetrad*

Let L_α be the propagation vector of a null electromagnetic field. In the absence of matter and with suitable units, the Ricci tensor of the space-time may be expressed as,

$$R_{\alpha\beta} = -L_\alpha L_\beta$$

The identity $R^{\alpha\beta}{}_{;\beta} = 0$ implies that $(L^\alpha L^\beta)_{;\beta} = 0$, and a vector L^α satisfying this relation also satisfies

$$\dot{L}_{[\alpha} L_{\beta]} = 0 \tag{2.1}$$

where $\dot{L}_\alpha = L^\gamma L_{\alpha;\gamma}$ (cf. Robinson, 1961).

Three other null vectors M^α, $\bar{M}^\alpha$, N^α can be constructed which, together with L^α, form a null tetrad. It has been shown (Goodinson & Newing, 1969) that the current vector J^α can be expressed as

$$J^\alpha = JL^\alpha + XM^\alpha + \bar{X}\bar{M}^\alpha \tag{2.2}$$

where J and X are quantities expressible in terms of the tetrad vectors. If $J_\alpha J^\alpha = 0$, the current is null, and in this case $X = 0$, i.e.

$$M^\alpha M^\beta L_{\alpha;\beta} = 0. \tag{2.3}$$

If (2.1) and (2.3) are satisfied, then L^α defines a shear-free family of geodesics (Robinson, 1961; Mariot, 1954; Sachs, 1961; Szekeres, 1966), and therefore L^α can be taken to be $L^\alpha = \lambda(dx^\alpha/dv)$, v being some parameter along the geodesic. Introducing a coordinate system in which the parameter v is taken to be the coordinate x^0, $L^\alpha = \lambda\delta_0{}^\alpha$, and since L^α is a null vector it follows that $g_{00} = 0$ in this coordinate system. The metric tensor of the space-time is given in terms of the tetrad by

$$g_{\alpha\beta} = 2L_{(\alpha} N_{\beta)} - 2M_{(\alpha} \bar{M}_{\beta)} \tag{2.4}$$

and now has the form

$$\begin{bmatrix} 0 & X & X & X \\ X & X & X & X \\ X & X & X & X \\ X & X & X & X \end{bmatrix}$$

The self-dual tensor $\mathring{\omega}^{\alpha\beta} = 2L^{[\alpha} M^{\beta]}$ can be constructed, and taking $\omega^{\alpha\beta} = \exp(i\theta)\mathring{\omega}^{\alpha\beta}$ to be the sum of the electromagnetic field tensor and its complex dual, Maxwell's equations for the electromagnetic field are (Goodinson & Newing, 1968)

$$[\exp(i\theta)\, \mathring{\omega}^{\alpha\beta}]_{;\beta} = J^\alpha$$

θ being a complexion parameter of the null field and J^α the current-density vector.

3. *The Exceptional Case*

Using the properties of the tetrad vectors as given in Goodinson & Newing (1969), it can be shown that if $(L^\alpha L^\beta)_{;\beta} = 0$, then

$$\tfrac{1}{2}\sqrt{g}\, \epsilon^{\alpha\beta\gamma\delta} L_\beta L_{\gamma;\delta} = \tfrac{1}{2}L^\alpha \bar{M}^\delta M^\gamma(L_{\gamma;\delta} - L_{\delta;\gamma})$$

and so the condition

$$\epsilon^{\alpha\beta\gamma\delta} L_\beta L_{\gamma;\delta} = 0 \qquad \text{(3.1) (Geroch condition)}$$

implies that

$$\bar{M}^\delta M^\gamma(L_{\gamma;\delta} - L_{\delta;\gamma}) = 0 \qquad \text{(3.2) (Peres condition)}$$

and both conditions imply that L_α is a scalar multiple of a gradient, $L_\alpha = \xi U_{,\alpha}$.

The possibility of setting up a standard form for the metric in the case of a charge-free null field whose propagation vector is a scalar multiple of a gradient was noticed by Geroch (1966). Wyman & Trollope (1965) discovered a metric of similar type. This method can also be used for the case of a non-zero charge-current distribution. With signature −2 and coordinates x^0, x^1, x^2, x^3 the Wyman–Trollope metric is

$$g_{\alpha\beta} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & \alpha & \beta & \gamma \\ 0 & \beta & -k & 0 \\ 0 & \gamma & 0 & -k \end{bmatrix}$$

The vacuum electromagnetic equations are automatically satisfied and the coefficients α, β, γ, k are to be chosen to satisfy the gravitational field equations, $R_{\mu\nu} = -L_\mu L_\nu$. Geroch's metric is the special case in which α, β and γ are independent of x^0, and $k = 1$. If $L_\alpha = \xi U_{,\alpha}$, then $U_0 = 0$, since $L_\alpha L^\alpha = 0$, and a coordinate system may be chosen in which $U = x^1$. In this case, L_α may be expressed in the form $L_\alpha = \lambda h \delta_\alpha{}^1$, and since $L^\alpha = \lambda \delta_0{}^\alpha$, it follows that $g_{0\mu} = h\delta_\mu{}^1$ and $g^{1\mu} = h^{-1}\delta_0{}^\mu$. The metric tensor and its reciprocal are therefore of the form

$$g_{\alpha\beta} = \begin{bmatrix} 0 & X & 0 & 0 \\ X & X & X & X \\ 0 & X & X & X \\ 0 & X & X & X \end{bmatrix}, \qquad g^{\alpha\beta} = \begin{bmatrix} X & X & X & X \\ X & 0 & 0 & 0 \\ X & 0 & X & X \\ X & 0 & X & X \end{bmatrix}$$

The condition $L_\alpha M^\alpha = 0$ implies that $M^1 = 0$, and by a suitable ψ-transformation (Peres, 1961) M^0 can be taken to be zero. M^α may then be expressed as

$$M^\alpha = a[\delta_2{}^\alpha + \rho \exp(i\eta)\,\delta_3{}^\alpha]$$

where ρ and η are real, and by a ϕ-transformation (Peres, 1961) a can also be taken to be real.

Since the contravariant tensor $g^{\alpha\beta}$ may be expressed in terms of the tetrad in the form

$$g^{\alpha\beta} = L^\alpha N^\beta + L^\beta N^\alpha - M^\alpha \bar{M}^\beta - \bar{M}^\alpha M^\beta$$

the 2×2 submatrix g^{mn}, $m, n = 2, 3$ is

$$g^{mn} = -2a^2 g'^{mn}$$

where

$$g'^{mn} = \begin{bmatrix} 1 & \rho\cos\eta \\ \rho\cos\eta & \rho^2 \end{bmatrix}$$

and the determinant g of the covariant tensor $g_{\alpha\beta}$ is $g = -h^2(4a^4\rho^2\sin^2\eta)^{-1}$. The electromagnetic field equations $[\bar{\omega}^{\alpha\beta}\exp(i\theta)]_{;\beta} = J^\alpha$, or equivalently $\{(L^\alpha M^\beta - L^\beta M^\alpha)\exp(i\theta)\sqrt{-g}\}_{,\beta} = J^\alpha\sqrt{-g}$, where J^α is real, lead to

$$\{a\lambda\sqrt{(-g)}.\exp(i\theta)\,\delta_0{}^\alpha[\delta_2{}^\beta + \rho\delta_3{}^\beta\exp(i\eta)]\}_{,\beta} - \{a\lambda\sqrt{(-g)}.\exp(i\theta)\,[\delta_2{}^\alpha + \rho\delta_3{}^\alpha\exp(i\eta)]\}_{,0} = \sqrt{(-g)}.J^\alpha \qquad (3.3)$$

From equation (2.2), since J^α is proportional to L^α, it follows that

$$\{a\lambda\sqrt{(-g)}.\exp(i\theta)\,[\delta_2{}^\alpha + \rho\delta_3{}^\alpha\exp(i\eta)]\}_{,0} = 0$$

and hence ρ and η are both independent of x^0.

The 2×2 submatrix g_{mn} of $g_{\alpha\beta}$ can be shown to be

$$g_{mn} = -(2a^2\rho^2\sin^2\eta)^{-1} g'_{mn}$$

$$g'_{mn} = \begin{bmatrix} \rho^2 & -\rho\cos\eta \\ -\rho\cos\eta & 1 \end{bmatrix}.$$

Since g'_{mn} is independent of x^0, it is possible to choose a coordinate transformation

$$x^0 = \bar{x}^0, \qquad x^1 = \bar{x}^1, \qquad x^2 = f(\bar{x}^1, \bar{x}^2, \bar{x}^3)$$
$$x^3 = g(\bar{x}^1, \bar{x}^2, \bar{x}^3)$$

which converts $g_{\alpha\beta}$ into the form

$$g_{\alpha\beta} = \begin{bmatrix} 0 & h & 0 & 0 \\ h & \alpha & \beta & \gamma \\ 0 & \beta & -k & 0 \\ 0 & \gamma & 0 & -k \end{bmatrix} \tag{3.4}$$

and by a further suitable transformation of coordinates, h can be taken to be unity.

The determinant g is then given by $g = -k^2$ and $g^{\alpha\beta}$ has the form

$$g^{\alpha\beta} = \begin{bmatrix} \rho & 1 & \beta/k & \gamma/k \\ 1 & 0 & 0 & 0 \\ \beta/k & 0 & -k^{-1} & 0 \\ \gamma/k & 0 & 0 & -k^{-1} \end{bmatrix}$$

where

$$\rho = -\left(\alpha + \frac{\beta^2}{k} + \frac{\gamma^2}{k}\right)$$

The tetrad vectors may then be taken to be

$$L_\alpha = \lambda \delta_\alpha{}^1, \qquad L^\alpha = \lambda \delta_0{}^\alpha$$

$$M_\alpha = \frac{1}{\sqrt{(2k)}}\{(\beta - i\gamma)\,\delta_\alpha{}^1 - k(\delta_\alpha{}^2 - i\delta_\alpha{}^3)\}$$

$$M^\alpha = \frac{1}{\sqrt{(2k)}}(\delta_2{}^\alpha - i\delta_3{}^\alpha)$$

$$N_\alpha = \lambda^{-1}\,\delta_\alpha{}^0 - \frac{\rho}{2\lambda}\,\delta_\alpha{}^1$$

$$N^\alpha = (\lambda k)^{-1}\{\tfrac{1}{2}\rho k \delta_0{}^\alpha + k\delta_1{}^\alpha + \beta\delta_2{}^\alpha + \gamma\delta_3{}^\alpha\}$$

the signs of the imaginary terms in M_α and M^α being chosen to ensure that $L^{[\alpha} M^{\beta]}$ is equal to its complex dual.

With the space-time defined by (3.4)

$$\omega^{\alpha\beta}{}_{;\beta} = \frac{\lambda \exp(i\theta)}{\sqrt{(2k)}}\{(\chi + i\theta)_{,2} - i(\chi + i\theta)_{,3}\}\,\delta_0{}^\alpha = J^\alpha$$

where $\chi = \log(\lambda\sqrt{k})$.

The condition that J^α be real may be satisfied by taking

$$\cos\theta = \exp(-\chi)\,F_{,2}, \qquad \sin\theta = \exp(-\chi)\,F_{,3}$$

where F is a function of x^1, x^2, x^3 subject to the condition $(F_{,2})^2 + (F_{,3})^2 = \lambda^2 k$, the current being given by the null vector

$$J^\alpha = \frac{1}{\sqrt{(2)}.k}(F_{,22} + F_{,33})\,\delta_0{}^\alpha$$

The parameters α, β, γ, k defining the space-time, and the electromagnetic parameter λ, must be such that the gravitational field equations are satisfied

$$R_{\alpha\beta} = -L_\alpha L_\beta = -\lambda^2\,\delta_\alpha{}^1\,\delta_\beta{}^1$$

A theorem may now be stated: The metric for any space-time admitting a null field with a null current can be expressed in the Wyman–Trollope form.

It may be noticed that in the case of zero charge-current, Wyman–Trollope's results follow with $\lambda\sqrt{k} = 1$, and Geroch's results with $k = 1$. In these cases, the electromagnetic field equations are satisfied by taking θ and χ to be independent of x^2 and x^3. Geroch's restriction that α, β, γ be independent of x^0 appears to be unnecessary, and arises from the assumption (Geroch, 1966, p. 173) that the vanishing of the trace of the square matrix D implies that D itself is zero; this is the case only if the matrix D is symmetric.

4. *The Vaidya Radiating Mass Particle*

As an example of the above theory, the substitutions $\alpha = 1 - (2m/x^0)$, $k = x^{02}\operatorname{sech}^2 x^2$, $\beta = 0 = \gamma$ are such that the only non-vanishing component of $R_{\alpha\beta}$ is $R_{11} = 2m_1/x^{02}$, where $m = m(x^1)$ and $m_1 \equiv dm/dx^1$. These substitutions have a physical interpretation (Goodinson, 1969) if one makes the further transformations $x^0 = r$, $x^1 = u$, $\tanh x^2 = \cos\theta$, $x^3 = \phi$. The metric $g_{\alpha\beta}$ of the space-time is then such that the line-element becomes

$$ds^2 = \left(1 - \frac{2m}{r}\right) du^2 + 2\,du\,dr - r^2(d\theta^2 + \sin^2\theta\,d\phi^2)$$

which represents the Vaidya (1947) spherically symmetric radiation field.

In the coordinate system $x^\mu = (x^0, x^1, x^2, x^3)$ the function F defined in Section 3 is such that

$$F_{,2}{}^2 + F_{,3}{}^2 = -2m_1\operatorname{sech}^2 x^2 \tag{4.1}$$

and the current-density vector J^μ is given by

$$J^\mu = \frac{\cosh^2 x^2}{\sqrt{(2)}.(x^0)^2}(F_{,22} + F_{,33})\,\delta_0{}^\mu \tag{4.2}$$

In the coordinate system $x^\mu = (r, u, \theta, \phi)$ equation (4.1) becomes

$$F_{,\theta}{}^2 + F_{,\phi}{}^2\operatorname{cosec}^2\theta = -2m_1 = \text{a function independent of } \theta \text{ and } \phi \tag{4.3}$$

and equation (4.2) gives

$$J^\mu = \frac{1}{\sqrt{(2)}.r^2}\cdot\frac{1}{\sin^2\theta}\left\{\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{\partial^2 F}{\partial\phi^2}\right\}\delta_0{}^\mu \tag{4.4}$$

Writing $F = (-2m_1)^{1/2} \Lambda(\theta, \phi)$, equation (4.3) becomes

$$\Lambda_{,\theta}^{2} + \Lambda_{,\phi}^{2} \operatorname{cosec}^2 \theta = 1 \tag{4.5}$$

and J^μ is given by

$$J^\mu = \frac{(-m_1)^{1/2}}{r^2} \nabla^2 \Lambda \delta_0{}^\mu$$

Equation (4.5) may be written as $|\nabla\Lambda|^2 = 1$, and it follows that $J^\mu \neq 0$.

Vaidya (1947) chose $\Lambda = \cos^{-1}\{\sin\theta \sin\phi\}$ as a solution of equation (4.5), and the current-density vector is then proportional to $\cot\Lambda$.

It is of interest to note that the simple solution of (4.5), i.e. $\Lambda = \theta$, represents the same physical situation as the Vaidya solution, if one just makes an interchange of the y- and z-axes of the coordinate system.

It would appear that a general solution of (4.5) can be expressed in the form $\Lambda = \cos^{-1}\{\hat{\mathbf{n}} . \hat{\mathbf{r}}\}$, where the unit vector $\hat{\mathbf{n}}$ defines a fixed direction and $r\hat{\mathbf{r}}$ is the position vector, i.e. Λ is the inclination of $\mathbf{r}$ to some fixed line in space.

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